Hamiltonian dynamics of dust-plasma interactions

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Hamiltonian dynamics of a collection of charged dust particles interacting with electric fields in the sheath region of a low-temperature plasma is studied. It is shown that the wake potential is formed behind a moving dust particle when virtual phonons are exchanged between dust grains moving relative to the ambient plasma, while the wake potential vanishes if the dust grains are stationary with respect to the plasma. The semiclassical Hamiltonian formulation is also applied to obtain dispersion relations for transverse and longitudinal oscillations associated with chains of dust grains levitated in the balance of gravitation and electric sheath field. [S1063-651X(98)05903-0]

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I. INTRODUCTION

Recently, the interest in Coulomb systems of charged particles has been greatly increased since the formation of Coulomb quasilattices, predicted theoretically by Ikezi [1], involving micrometer sized highly charged dust particulates, has been successfully demonstrated experimentally [2]. In the experiments, the crystal-like structures are observed in the sheath region of a low-temperature radio-frequency gas discharge plasma where there is balance between the gravitational and electrostatic forces. The distance between the dust grains is of order plasma Debye length λ_D.

Two- and three-dimensional structures as well as phase transitions observed in the dust-plasma systems [3–7] require adequate theoretical description. One of the most favored is the Hamiltonian formalism since it is very useful in statistical and interaction analyses as well as in numerical simulations [8]. The knowledge of the system's Hamiltonian is essential for a description of propagation and interaction of modes associated with the dust motion and especially for the calculation of the free energy to study phase transitions and critical phenomena.

In this paper, we derive the semiclassical Hamiltonian, which describes the interaction with external fields, the screened Coulomb potential, and the effective interaction of dust particulates by exchanging virtual phonons. The study of the effective interaction is motivated by the finding of an oscillating stationary wake behind a static test particle [9–11] in the sheath region where strong ion flow exceeding the ion acoustic velocity is established [12,13]. The interaction of the particles in the wake field is similar to the Cooper pairing of electrons in superconductors [14], and was earlier studied, e.g., for two-component electron-ion plasmas [15]. Furthermore, we demonstrate how the oscillations of the dust grains, viz. longitudinal [16] and ''transverse'' [17] lattice modes, can be obtained within the framework of the present formalism. The excitation of the vibrational modes, especially of the ''transverse'' waves due to vertical oscillations of the grains, may be responsible for phase transitions in the system.

II. HAMILTONIAN OF THE SYSTEM

We consider a collection of charged test particles (dust particulates), with the coordinate $\mathbf{x}_j$ and the momentum $\mathbf{p}_j$, embedded in a background plasma and interacting with longitudinal collective plasma fields. The kinetic energy of the test particles is given by the particle Hamiltonian $\Sigma_j p_j^2/2m_j$. The interaction of test particles with the longitudinal electric fields $\mathbf{E}(\mathbf{x},t)$ may be conveniently expressed in terms of the longitudinal vector potential $\mathbf{A}$ by replacing $\mathbf{p}_j$ by $\mathbf{p}_j - (Z_j e/c)\mathbf{A}(\mathbf{x}_j)$ in the Hamiltonian. A collection of test particles interacts not only with the longitudinal electric fields, but also through forces derivable from the external potential V_{ext} such as a sheath potential. The Hamiltonian for our system is thus given by [18]

$$H=\sum_j \frac{1}{2m_j}\left[\mathbf{p}_j-\frac{Z_j e}{c}\mathbf{A}(\mathbf{x}_j,t)\right]^2+\int d\mathbf{x}\frac{\mathbf{E}^2}{8\pi}+V_{\text{ext}}, \quad (1)$$

where the summation is over the test particles with masses m_j, momenta $\mathbf{p}_j$, and charges $Z_j e$, $\mathbf{A}(\mathbf{x},t)$ is the longitudinal vector potential (in the assumed gauge the scalar potential ϕ is zero), and $\mathbf{E}(\mathbf{x},t)$ is the longitudinal electric field. We introduce

$$\mathbf{A}(\mathbf{x},t)=\sum_{\mathbf{k}}\left[\frac{4\pi c^2}{V|\mathbf{k}|^2}\right]^{1/2} q_{\mathbf{k}}(t)\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (2)$$

where V is the volume of the system; then the electric field can be written as

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$$\mathbf{E}(\mathbf{x},t)=-\frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t}=-\sum_{\mathbf{k}}\left[\frac{4\pi}{V|\mathbf{k}|^2}\right]^{1/2}\dot{q}_{\mathbf{k}}(t)\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3}$$

with $\dot{q}_{\mathbf{k}}(t)\equiv dq_{\mathbf{k}}(t)/dt=-\wp_{-\mathbf{k}}$. Thus we separate the Hamiltonian (1) as

$$H=H_{\mathbf{p}}+H_f+H_I^{(1)}+H_I^{(2)}+V_{\mathrm{ext}}, \tag{4}$$

where the kinetic energy of the particles is given by

$$H_{\mathbf{p}}=\sum_j\frac{1}{2m_j}\mathbf{p}_j^2, \tag{5}$$

the energy of the electric field is

$$H_f=\int d\mathbf{x}\frac{\mathbf{E}^2}{8\pi}=-\frac{1}{2}\sum_{\mathbf{k}}\wp_{\mathbf{k}}\wp_{-\mathbf{k}}, \tag{6}$$

and the interaction terms are given by

$$H_I^{(1)}=-\sum_j\sum_{\mathbf{k}}\frac{Z_je}{m_j}\left[\frac{4\pi}{V|\mathbf{k}|^2}\right]^{1/2}\mathbf{k}\cdot\left(\mathbf{p}_j-\frac{\hbar\mathbf{k}}{2}\right)q_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}_j} \tag{7}$$

and

$$H_I^{(2)}=\sum_j\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\frac{2\pi Z_j^2e^2\mathbf{k}\cdot\mathbf{k}'}{m_jV|\mathbf{k}||\mathbf{k}'|}q_{\mathbf{k}}q_{\mathbf{k}'}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}_j}. \tag{8}$$

The short-range Coulomb static interaction term can be explicitly obtained from Eq. (4) by applying a unitary transformation

$$U=\exp\left\{-\frac{1}{h}\sum_j\sum_{|\mathbf{k}|>\lambda_D^{-1}}\left[\frac{4\pi Z_j^2e^2}{V|\mathbf{k}|^2\varepsilon(\mathbf{k},0)}\right]^{1/2}q_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}_j}\right\}, \tag{9}$$

where the $\mathbf{k}$ summation is restricted for $|\mathbf{k}|>\lambda_D^{-1}$, $\varepsilon(\mathbf{k},0)=1+|\mathbf{k}|^{-2}\lambda_D^{-2}$ is the static plasma form factor, and λ_D is the plasma Debye length; in the case of only plasma electron contribution we have $\lambda_D=\lambda_{De}=(T_e/4\pi n_ee^2)^{1/2}$, where T_e, n_e, and $-e$ is the electron temperature, density, and charge, respectively. Thus we find

$$H\to U^{-1}HU=H_{\mathbf{p}}+H_f+H_I^{(1)}+H_I^{(2)}+V_{\mathrm{ext}}+\frac{1}{2}\sum_{|\mathbf{k}|>\lambda_D^{-1}}\sum_{i\neq j}\frac{4\pi e^2Z_iZ_je^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}}{V|\mathbf{k}|^2\varepsilon(\mathbf{k},0)}, \tag{10}$$

where $H_{\mathbf{p}}$, H_f, $H_I^{(1)}$, and $H_I^{(2)}$ are given by Eqs. (5) to (8) with the $\mathbf{k}$ summation restricted to $|\mathbf{k}|<\lambda_D^{-1}$. We neglected terms including the factor $\{1-1/[\varepsilon(\mathbf{k},0)]^{1/2}\}$ for $|\mathbf{k}|>\lambda_D^{-1}$ since the Debye screening makes the plasma wave propagation impossible in the wave number range $|\mathbf{k}|>\lambda_D^{-1}$ (the effects of dynamic screening will be included below). Such a procedure of the transformation introduces the wave number domain appropriate for the interactions in consideration, and it is similar to that invoked in Ref. [18].

We can now write the Hamiltonian appropriate to a set of harmonic oscillators, which represent the collective field of the plasma wave as

$$H_{\mathrm{osc}}=H_f+H_I^{(2)}=\frac{1}{2}\sum_{\mathbf{k}}\left(\wp_{\mathbf{k}}\wp_{\mathbf{k}}^*+\sum_j\omega_{pj}^2q_{\mathbf{k}}q_{\mathbf{k}}^*\right), \tag{11}$$

where we have taken $\mathbf{k}=-\mathbf{k}'$ for $H_I^{(2)}$. Here, $\wp_{\mathbf{k}}^*=-\wp_{-\mathbf{k}}$ and $q_{\mathbf{k}}^*=-q_{-\mathbf{k}}$ because of the reality condition for the electric field, and $\omega_{pj}=(4\pi Z_j^2e^2/Vm_j)^{1/2}$ is the frequency of the collective particle motion. Taking into account the dynamical screening of the wave fields and setting

$$\wp_{\mathbf{k}}=i\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{(\partial\omega\varepsilon/\partial\omega)_{\omega_{\mathbf{k}}}}}(a_{-\mathbf{k}}+a_{\mathbf{k}}^*),$$

$$q_{\mathbf{k}}=\sqrt{\frac{\hbar}{\omega_{\mathbf{k}}(\partial\omega\varepsilon/\partial\omega)_{\omega_{\mathbf{k}}}}}(a_{\mathbf{k}}-a_{-\mathbf{k}}^*), \tag{12}$$

we obtain

$$H_I^{(1)}=-\sum_j\sum_{\mathbf{k}}\frac{Z_je}{m_j}\left[\frac{4\pi\hbar}{V|\mathbf{k}|^2\omega_{\mathbf{k}}(\partial\omega\varepsilon/\partial\omega)_{\omega_{\mathbf{k}}}}\right]^{1/2}\times\left[\mathbf{k}\cdot\left(\mathbf{p}_j-\frac{\hbar\mathbf{k}}{2}\right)a_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}_j}+e^{-i\mathbf{k}\cdot\mathbf{x}_j}a_{\mathbf{k}}^*(t)\mathbf{k}\cdot\left(\mathbf{p}_j-\frac{\hbar\mathbf{k}}{2}\right)\right] \tag{13}$$

as well as

$$H_{\mathrm{osc}}=\sum_{\mathbf{k}}\frac{\hbar\omega_{\mathbf{k}}}{(\partial\omega\varepsilon/\partial\omega)_{\omega_{\mathbf{k}}}}(a_{\mathbf{k}}^*a_{\mathbf{k}}+a_{\mathbf{k}}a_{\mathbf{k}}^*)+\sum_{\mathbf{k}}\frac{\hbar}{2\omega_{\mathbf{k}}(\partial\omega\varepsilon/\partial\omega)_{\omega_{\mathbf{k}}}}\left(\sum_j\omega_{pj}^2-\omega_{\mathbf{k}}^2\right)\times(a_{\mathbf{k}}^*a_{\mathbf{k}}+a_{\mathbf{k}}a_{\mathbf{k}}^*-a_{\mathbf{k}}a_{-\mathbf{k}}-a_{-\mathbf{k}}^*a_{\mathbf{k}}^*), \tag{14}$$

where $\varepsilon=\varepsilon(\mathbf{k},\omega)$ is the linear plasma dielectric permittivity. Solution of the dispersion equation $\varepsilon(\mathbf{k},\omega)=0$ gives the eigenfrequency of the plasma waves $\omega=\omega_{\mathbf{k}}$. Below, we assume that there is no plasma wave damping, and consider the positive wave eigenfrequencies, $\omega_{\mathbf{k}}>0$.

Next, we apply the canonical transformations

$$\mathbf{x}_j=e^{-(i/\hbar)S}\mathbf{X}_j e^{(i/\hbar)S}, \quad \mathbf{p}_j=e^{-(i/\hbar)S}\mathbf{P}_j e^{(i/\hbar)S}, \ldots, \tag{15}$$

where

$$S=i\sum_j \sum_{\mathbf{k}} (\alpha_{j\mathbf{k}}A_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{X}_j}-e^{-i\mathbf{k}\cdot\mathbf{X}_j}A_{\mathbf{k}}^{\dagger}\alpha_{j\mathbf{k}}) \tag{16}$$

and

$$\alpha_{j\mathbf{k}}=\frac{Z_j e}{m_j}\left[\frac{4\pi\hbar}{V|\mathbf{k}|^2\omega_{\mathbf{k}}(\partial\omega\varepsilon/\partial\omega)_{\omega_{\mathbf{k}}}}\right]^{1/2} \times\frac{\mathbf{k}\cdot(\mathbf{P}_j-\hbar\mathbf{k}/2)}{\omega_{\mathbf{k}}-(\mathbf{k}\cdot\mathbf{P}_j/m_j)+(\hbar|\mathbf{k}|^2/2m_j)} \tag{17}$$

to obtain the set of new variables $(\mathbf{X}_j,\mathbf{P}_j,A_{\mathbf{k}},A_{\mathbf{k}}^{\dagger},\mathcal{H})$ from $(\mathbf{x}_j,\mathbf{p}_j,a_{\mathbf{k}},a_{\mathbf{k}}^{*},H)$. We have

$$\mathbf{p}_j=\mathbf{P}_j+\left(-\frac{i}{\hbar}\right)[S,\mathbf{P}_j]+\frac{1}{2}\left(-\frac{i}{\hbar}\right)^2[S,[S,\mathbf{P}_j]]+\cdots$$
$$=\mathbf{P}_j-\sum_{\mathbf{k}}\mathbf{k}(\alpha_{j\mathbf{k}}A_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{X}_j}+e^{i\mathbf{k}\cdot\mathbf{X}_j}A_{\mathbf{k}}^{\dagger}\alpha_{j\mathbf{k}})+\cdots, \tag{18}$$

$$a_{\mathbf{k}}=A_{\mathbf{k}}+\left(-\frac{i}{\hbar}\right)[S,A_{\mathbf{k}}]+\frac{1}{2}\left(-\frac{i}{\hbar}\right)^2[S,[S,A_{\mathbf{k}}]]+\cdots$$
$$=A_{\mathbf{k}}+\frac{1}{\hbar}\sum_j e^{-i\mathbf{k}\cdot\mathbf{X}_j}\alpha_{j\mathbf{k}}+\cdots, \tag{19}$$

$$a_{\mathbf{k}}^{*}=A_{\mathbf{k}}^{\dagger}+\left(-\frac{i}{\hbar}\right)[S,A_{\mathbf{k}}^{\dagger}]+\frac{1}{2}\left(-\frac{i}{\hbar}\right)^2[S,[S,A_{\mathbf{k}}^{\dagger}]]+\cdots$$
$$=A_{\mathbf{k}}^{\dagger}+\frac{1}{\hbar}\sum_j \alpha_{j\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{X}_j}+\cdots, \tag{20}$$

and

$$e^{i\mathbf{k}\cdot\mathbf{x}_j}=e^{i\mathbf{k}\cdot\mathbf{X}_j}-\left(-\frac{i}{\hbar}\right)[S,e^{i\mathbf{k}\cdot\mathbf{X}_j}]+\frac{1}{2}\left(-\frac{i}{\hbar}\right)^2[S,[S,e^{i\mathbf{k}\cdot\mathbf{X}_j}]]+\cdots$$
$$=e^{i\mathbf{k}\cdot\mathbf{X}_j}-\sum_{\mathbf{k}'}\left(\frac{4\pi Z_j^2e^2\omega_{\mathbf{k}}}{V\hbar|\mathbf{k}'|^2(\partial\omega\varepsilon/\partial\omega)_{\omega_{\mathbf{k}'}}}\right)^{1/2}\left[\left(\frac{1}{\omega_{\mathbf{k}'}-(\mathbf{k}'\cdot\mathbf{P}_j/m_j)+(\hbar|\mathbf{k}|'^2/2m_j)}\right.\right.$$
$$\left.-\frac{1}{\omega_{\mathbf{k}'}-(\mathbf{k}'\cdot\mathbf{P}_j/m_j)+(\hbar|\mathbf{k}'|^2/2m_j)+(\hbar\mathbf{k}'\cdot\mathbf{k}/m_j)}\right)A_{\mathbf{k}'}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{X}_j}$$
$$\left.+A_{\mathbf{k}'}^{\dagger}e^{-i\mathbf{k}'\cdot\mathbf{X}_j}\left(\frac{1}{\omega_{\mathbf{k}'}-(\mathbf{k}'\cdot\mathbf{P}_j/m_j)+(\hbar|\mathbf{k}'|^2/2m_j)+(\hbar\mathbf{k}\cdot\mathbf{k}'/m_j)}-\frac{1}{\omega_{\mathbf{k}'}-(\mathbf{k}'\cdot\mathbf{P}_j/m_j)+(\hbar|\mathbf{k}'|^2/2m_j)}\right)e^{i\mathbf{k}\cdot\mathbf{X}_j}\right]+\cdots, \tag{21}$$

where $[A,B]\equiv AB-BA$ is the commutator of A and B. We note that the $\mathbf{k}$ summation is restricted for $|\mathbf{k}|<\lambda_D^{-1}$ unless otherwise specified. It is straightforward to show that

$$\sum_j \frac{p_j^2}{2m_j}+\sum_{\mathbf{k}}\frac{\hbar\omega}{2}(a_{\mathbf{k}}^{*}a_{\mathbf{k}}+a_{\mathbf{k}}a_{\mathbf{k}}^{*})=\sum_j\frac{P_j^2}{2m_j}+\sum_{\mathbf{k}}\frac{\hbar\omega}{2}(A_{\mathbf{k}}^{\dagger}A_{\mathbf{k}}+A_{\mathbf{k}}A_{\mathbf{k}}^{\dagger})$$
$$+\sum_{j\mathbf{k}}\frac{Z_je_j}{m_j}\left(\frac{4\pi\hbar}{V|\mathbf{k}|^2\omega_{\mathbf{k}}(\partial\omega\varepsilon/\partial\omega)\omega_{\mathbf{k}}}\right)^{1/2}\left[\mathbf{k}\cdot\left(\mathbf{P_j}-\frac{\hbar\mathbf{k}}{\mathbf{2}}\right)A_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{X}_j}+e^{-i\mathbf{k}\cdot\mathbf{X}_j}A_{\mathbf{k}}^{\dagger}\mathbf{k}\cdot\left(\mathbf{P}_j-\frac{\hbar\mathbf{k}}{2}\right)\right]. \tag{22}$$

Our new canonically transformed interaction Hamiltonian includes the static Debye contribution $\mathcal{H}_D$, the external potential $\mathcal{V}_{\mathrm{ext}}$, the term $\mathcal{H}_I^{(1)}$ corresponding to $H_I^{(1)}$, and the interaction parts of $\mathcal{H}_{\mathbf{P}}$ and $\mathcal{H}_{\mathrm{osc}}$ in the lowest order, i.e.,

$$\mathcal{H}_{\mathrm{int}}=\mathcal{H}_D+\mathcal{V}_{\mathrm{ext}}+\mathcal{H}_I^{(1)}+\left(\mathcal{H}_{\mathbf{P}}-\sum_j\frac{\mathbf{P}_j^2}{2m_j}\right)+\left[\mathcal{H}_{\mathrm{osc}}-\sum_{\mathbf{k}}\frac{\hbar\omega_{\mathbf{k}}}{(\partial\omega\varepsilon/\partial\omega)_{\omega_{\mathbf{k}}}}(A_{\mathbf{k}}^{\dagger}A_{\mathbf{k}}+A_{\mathbf{k}}A_{\mathbf{k}}^{\dagger})\right], \tag{23}$$

where

$$\mathcal{H}_D=\frac{1}{2}\sum_{|\mathbf{k}|>\lambda_D^{-1}}\sum_{i\neq j}\frac{4\pi e^2 Z_i Z_j e^{i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)}}{V|\mathbf{k}|^2\varepsilon(\mathbf{k},0)}. \tag{24}$$

The last three terms in Eq. (23) will be combined and expressed as

$$\mathcal{V}^{\text{eff}}=-\sum_{i\neq j}\sum_{\mathbf{k}}\frac{2\pi Z_i Z_j e^2}{Vm_i m_j|\mathbf{k}|^2\omega_{\mathbf{k}}(\partial\omega\varepsilon/\partial\omega)_{\omega_{\mathbf{k}}}}\left[\mathbf{k}\cdot\left(\mathbf{P}_i-\frac{\hbar\mathbf{k}}{2}\right)\mathbf{k}\cdot\left(\mathbf{P}_j+\frac{\hbar\mathbf{k}}{2}\right)\frac{1}{\omega_{\mathbf{k}}-(\mathbf{k}\cdot\mathbf{P}_j/m_j)-(\hbar|\mathbf{k}|^2/2m_j)}e^{i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)}\right.$$
$$\left.+e^{-i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)}\mathbf{k}\cdot\left(\mathbf{P}_j+\frac{\hbar\mathbf{k}}{2}\right)\frac{1}{\omega_{\mathbf{k}}-(\mathbf{k}\cdot\mathbf{P}_j/m_j)-(\hbar|\mathbf{k}|^2/2m_j)}\mathbf{k}\cdot\left(\mathbf{P}_i-\frac{\hbar\mathbf{k}}{2}\right)\right]. \tag{25}$$

Here, the effective interaction between particles through the exchange of virtual plasma waves or quasiparticles is described. We note that the dynamical collective effect of background plasma is included through the dielectric function $\varepsilon(\mathbf{k},\omega)$. Below, we discuss some of the situations described by this interaction Hamiltonian.

III. EFFECTIVE POTENTIAL

Consider a pair of dust particles interacting via the plasma ion-acoustic waves. Within the acoustic frequency range, the plasma dielectric permittivity is given by

$$\varepsilon(\mathbf{k},\omega)=1+1/|\mathbf{k}|^2\lambda_D^2-\omega_{pi}^2/\omega^2, \tag{26}$$

where $\lambda_D=\lambda_{\text{De}}$ is the electron Debye length and $\omega_{pi}=(4\pi n_i e^2/m_i)^{1/2}$ is the ion plasma frequency (m_i, n_i, and e are the ion mass, density, and charge, respectively). Solution of the dispersion equation $\varepsilon(\mathbf{k},\omega)=0$ gives the dispersion of the plasma ion-acoustic mode

$$\omega_{\mathbf{k}}=|\mathbf{k}|C_s/\sqrt{1+|\mathbf{k}|^2\lambda_D^2}, \tag{27}$$

where $C_s=(T_e/m_i)^{1/2}$ is the ion acoustic velocity. Let a pair of dust grains have masses m_1,m_2 and charges $Q_1=Z_1e,Q_2=Z_2e$, and consider the quasiclassical limit $\hbar\to0$. When the particles move in the same direction Z with velocities v_1 and v_2, we find that the effective potential energy derived from the interaction Hamiltonian is given by

$$\mathcal{V}_{12}^{\text{eff}}=-\sum_{|\mathbf{k}|<\lambda_D^{-1}}\frac{4\pi Q_1Q_2k_Z^2v_1v_2}{V(|\mathbf{k}|^2+\lambda_D^{-2})}\left(\frac{1}{\omega_{\mathbf{k}}^2-k_Z^2v_1^2}+\frac{1}{\omega_{\mathbf{k}}^2-k_Z^2v_2^2}\right)e^{i\mathbf{k}\cdot(\mathbf{X}_1-\mathbf{X}_2)}. \tag{28}$$

Letting $V\to\infty$ and $\Sigma_{\mathbf{k}}\to V\int d^3k/(2\pi)^3=V[\int d\mathbf{k}_\perp/(2\pi)^2]\int dk_Z/2\pi$, we find

$$\mathcal{V}_{12}^{\text{eff}}=-\frac{Q_1Q_2v_1v_2}{(2\pi)^2}\int dk_Zd\mathbf{k}_\perp\frac{k_Z^2\lambda_D^2\exp(i\mathbf{k}\cdot\mathbf{R})}{1+|\mathbf{k}|^2\lambda_D^2}\left[\frac{1}{\omega_k^2-(k_Zv_2)^2}+\frac{1}{\omega_k^2-(k_Zv_1)^2}\right], \tag{29}$$

where the integration is limited in the range $|\mathbf{k}|<\lambda_D^{-1}$, $\mathbf{R}=\mathbf{X}_1-\mathbf{X}_2$, and $k^2=k_Z^2+|\mathbf{k}_\perp|^2$. We see that

$$\mathcal{V}_{12}^{\text{eff}}=0 \tag{30}$$

for $v_1=0$ or $v_2=0$ and

$$\mathcal{V}_{12}^{\text{eff}}=Q_1Q_2\int\frac{dk_Zd\mathbf{k}_\perp}{2\pi^2}\frac{\lambda_D^2\exp(i\mathbf{k}\cdot\mathbf{R})}{1+k^2\lambda_D^2}\left[1+\frac{\omega_k{}^2}{k_Z^2v_0^2-\omega_k{}^2}+\frac{1}{2}\left(\frac{\delta v}{v_0}\right)^2\left(1+\frac{\omega_k{}^2(3k_z^2v_0^2-\omega_k^2)}{(\omega_k^2-k_z^2v_0^2)^2}\right)\right] \tag{31}$$

for $v_1=v_0,v_2=v_0\pm\delta v(|\delta v|\ll|v_0|)$. In the limit of $\delta v\to0$, we obtain

$$\mathcal{V}_{12}^{\text{eff}}=Q_1Q_2\int\frac{dk_Zd\mathbf{k}_\perp}{2\pi^2}\frac{\lambda_D^2\exp(i\mathbf{k}\cdot\mathbf{R})}{1+k^2\lambda_D^2}\left[1+\frac{k^2\lambda_D^{-2}M^{-2}}{(k_Z^2+k_0^2)(k_Z^2-k_1^2)}\right], \tag{32}$$

where $M=|v_0|/C_s$ is the Mach number and $k_{0,1}^2=\pm[(1-M^{-2})\lambda_D^{-2}+|\mathbf{k}_\perp|^2]/2+\{|\mathbf{k}_\perp|^2M^{-2}\lambda_D^{-2}+[(1-M^{-2})\lambda_D^{-2}+|\mathbf{k}_\perp|^2]^2/4\}^{1/2}$. The vanishing effective potential was investigated in the context of the study of molecular-ion beams interacting with metals [19]. The effective potential, Eq. (32), is in agreement with the wake potential derived for the screened electrostatic potential due to a test dust particle in the ion flow [11]. It is noteworthy that the effective potential vanishes when one dust grain is stationary while another dust grain is moving with respect to the ambient plasma. The exchange of phonons between a pair of dust grains is only possible when a pair of dust grains is moving together (or alternatively they are stationary in the presence of plasma flow). The earlier work on the wake potential [9–11] could not reveal such a situation, since the wake potential was calculated by a screened potential due to a single test dust particle without the presence of neighboring dust particles. The Hamiltonian dynamics is thus essential to find the wake potential if dust particles move with different velocities.

Note that the addition of the first term in the square brackets in Eq. (32) and $\mathcal{H}_D$, given by Eq. (24), forms the complete Debye static interaction potential $\mathcal{V}_D=\Sigma_{i\neq j}\mathcal{V}_{ij}^{D}/2$, where

$$\mathcal{V}_{ij}^{D}=\sum_{\mathbf{k}}\frac{4\pi Q_iQ_je^{i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)}}{V|\mathbf{k}|^2\varepsilon(\mathbf{k},0)}. \tag{33}$$

Here, we note that the summation is within the whole range of possible values of $\mathbf{k}$. For $M>1$, the contribution from the poles at $\pm k_1$ in the k_Z plane gives the oscillatory wake potential [9–11] while the poles at $k_Z=\pm ik_0$ provide the nonoscillating part, which modifies the static Debye shielding scale λ_D for moving particles.

Assuming a cylindrical symmetry $\mathbf{k}\cdot\mathbf{R}=|\mathbf{k}_\perp|R_\perp\cos\phi+k_ZZ$ and integrating over k_Z in Eq. (32) we find the approximate expression for the oscillatory interaction potential energy at the distance $R_\perp>\lambda_D$ and $|Z|>\lambda_D\sqrt{M^2-1}$ as given by

$$\mathcal{V}_{12}^{W}(R_\perp,Z)\simeq\frac{2Q_1Q_2}{1-M^{-2}}\sqrt{\frac{\lambda_D}{2\pi R_\perp}}\left\{\frac{\cos[(\pi/4)+(Z_-/\lambda_D\sqrt{M^2-1})]}{Z_-}+\frac{\cos[(\pi/4)-(Z_+/\lambda_D\sqrt{M^2-1})]}{Z_+}\right\}, \tag{34}$$

where $Z_\pm\equiv|Z|\pm R_\perp\sqrt{M^2-1}>0$. We note that the oscillating potential exists only in the wake of the test particle, i.e., for $Z<0$ and $|Z|>R_\perp\sqrt{M^2-1}$ [10,11]. On the other hand, for $R_\perp<\lambda_D$ and $|Z|>\lambda_D\sqrt{M^2-1}$, we recover [9]

$$\mathcal{V}_{12}^{W}(R_\perp=0,Z)\simeq\frac{2Q_1Q_2}{|Z|}\frac{\cos(|Z|/\lambda_D\sqrt{M^2-1})}{1-M^{-2}}. \tag{35}$$

The Debye static interaction potential given by Eq. (33) can be expressed explicitly as

$$\mathcal{V}_{ij}^{D}=\frac{Q_iQ_j}{|\mathbf{X}_i-\mathbf{X}_j|}e^{-|\mathbf{X}_i-\mathbf{X}_j|/\lambda_D}. \tag{36}$$

IV. LATTICE OSCILLATIONS

Consider oscillations of the dust particles interacting via the potential $\mathcal{H}_{\rm int}$. Suppose dust particles of charge Q are levitating with the balance of gravity and the sheath electric field in the vertical Z direction. For linear modes, their longitudinal and transverse vibrations are decoupled and can be considered separately. In the case of the vertical vibrations of the particles in the Z direction, we have to take into account the potential well appearing as a result of the gravity and the sheath electric field. For the oscillations in a horizontal one-dimensional chain, we take into account the Debye and external potentials. The equation of motion is given by

$$\dot{P}_{j,Z}=-\frac{\partial}{\partial Z_j}\left(\sum_i\mathcal{V}_{ij}^{D}+\mathcal{V}_j^{\rm ext}\right), \tag{37}$$

where the dot denotes the time derivative, $\mathcal{V}_{ij}^{D}$ is given by Eq. (36), and $\mathcal{V}_j^{\rm ext}$ is the external potential resulting from the gravity and the sheath electric field and may be given, in the parabolic approximation, as

$$\mathcal{V}_j^{\rm ext}=\frac{1}{2}\gamma(\delta Z_j)^2. \tag{38}$$

Here, δZ_j is the small deviation from the equilibrium and γ is the constant defined by the slope of the sheath electric field at the equilibrium position of dust particles [17]. If we consider interaction only with the next neighboring particles, i.e., $i=j-1$ and $j+1$, the equation of motion becomes

$$\dot{P}_{j,Z}=\frac{QE(r_0)}{r_0}(2\delta Z_j-\delta Z_{j-1}-\delta Z_{j+1})-\gamma\delta Z_j, \tag{39}$$

where

$$E(r_0)=\frac{Q}{r_0^2}\left(1+\frac{r_0}{\lambda_D}\right)e^{-r_0/\lambda_D} \tag{40}$$

and $r_0=|\mathbf{X}_j-\mathbf{X}_{j-1}|=|\mathbf{X}_{j+1}-\mathbf{X}_j|$. Noting that

$$\dot{P}_{j,Z}=M_d\frac{d^2}{dt^2}\delta Z_j, \tag{41}$$

where M_d is the mass of the dust grain and setting

$$\delta Z_j=\delta Z_0e^{-i(\omega t-jkr_0)}, \tag{42}$$

we find the dispersion relation for the transverse oscillation of horizontal lattice chain as

$$\omega^2=\frac{\gamma}{M_d}-\frac{4Q^2}{M_dr_0^3}\left(1+\frac{r_0}{\lambda_D}\right)\exp\left(-\frac{r_0}{\lambda_D}\right)\sin^2\left(\frac{kr_0}{2}\right). \tag{43}$$

This is the dispersion relation for the transverse oscillation of the horizontal lattice chain. A detailed study on this dispersion relation is reported by Vladimirov *et al.* [17].

In the case of longitudinal motions in the arrangement of a horizontal one-dimensional chain, their oscillations are governed by a Hamilton equation,

$$\dot{P}_{j,X}=-\frac{\partial}{\partial X_j}\sum_i\mathcal{V}_{ij}^{D}, \tag{44}$$

which, after taking the relevant neighbor interactions as for the transverse oscillation, becomes

$$\dot{P}_{j,X}=Q\left(\frac{\partial E}{\partial r}\right)_{r_0}(2\,\delta X_j-\delta X_{j-1}-\delta X_{j+1}), \tag{45}$$

where

$$\left(\frac{\partial E}{\partial r}\right)_{r_0}=-\frac{2Q}{r_0^3}\left(1+\frac{r_0}{\lambda_D}+\frac{r_0^2}{2\lambda_D^2}\right)e^{-r_0/\lambda_D}. \tag{46}$$

Noting that

$$\dot{P}_{j,X}=M_d\frac{d^2}{dt^2}\delta X_j\,, \tag{47}$$

and setting

$$\delta X_j=\delta X_0 e^{-i(\omega t-jkr_0)}, \tag{48}$$

we obtain the dispersion relation for the longitudinal oscillation as

$$\omega^2=\frac{2Q^2}{M_d r_0^3}\left(1+\frac{r_0}{\lambda_D}+\frac{r_0^2}{2\lambda_D^2}\right)\exp\left(-\frac{r_0}{\lambda_D}\right)\sin^2\left(\frac{kr_0}{2}\right), \tag{49}$$

which agrees with the equation derived by Melandsø by the Green's function method [16].

Next consider the case of a one-dimensional vertical chain when particles are placed one above another. A vertical equilibrium of N particles will be established in the chain. Recently the vertical chain of dust particles has been observed in the experiments in [20]. The vertical motion in the vertical arrangement of three dust particles separated by the equal distance is, taking only nearest-neighbor interactions and assuming no ion flows, described by

$$\dot{P}_{1,Z}=-\gamma\delta Z_1-\frac{\partial\mathcal{V}_{12}^{D}}{\partial Z},$$

$$\dot{P}_{2,Z}=-\gamma\delta Z_2-\frac{\partial\mathcal{V}_{21}^{D}}{\partial Z}-\frac{\partial\mathcal{V}_{23}^{D}}{\partial Z}, \tag{50}$$

$$\dot{P}_{3,Z}=-\gamma\delta Z_3-\frac{\partial\mathcal{V}_{32}^{D}}{\partial Z}.$$

In the linear approximation, expanding Eq. (50) in small perturbations of the equilibrium and taking into account the equal separation of the grains, we find the dispersion equation

$$(-\omega^2 M_d+\gamma)[(-\omega^2 M_d+\gamma+2\gamma_D)^2-\gamma_D^2]=0, \tag{51}$$

where

$$\gamma_D=\left|\frac{\partial^2\mathcal{V}_{12}^{D}}{\partial Z^2}\right|\approx\left|\frac{\partial^2\mathcal{V}_{23}^{D}}{\partial Z^2}\right|. \tag{52}$$

Thus we obtain the three characteristic frequencies of oscillations in the vertical chain of three dust grains

$$\omega_1^2=\frac{\gamma}{M_d},\quad \omega_2^2=\frac{1}{M_d}(\gamma+\gamma_D),\quad \omega_3^2=\frac{1}{M_d}(\gamma+3\gamma_D). \tag{53}$$

Here, the first frequency corresponds to all three particles moving together with equal amplitudes, and the second and third frequencies correspond to different relations between phases and amplitudes of the particle vibrations.

V. CONCLUSIONS

In summary, we have derived the semiclassical interaction Hamiltonian for the ensemble of dust particles in a plasma. The Hamiltonian describes the effective potential produced by a pair of moving dust grains due to their interaction with the external fields, Debye-Hückel screening potential, and the exchange of virtual phonons in the ion flow. It is shown that the wake potential will vanish if one of the pair of dust particles is stationary with respect to the ambient plasma. The Hamiltonian is also applied to oscillations in a coupled system of dust particles, and characteristic frequencies of both longitudinal and transverse modes are derived.

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[1] H. Ikezi, Phys. Fluids **29**, 1764 (1986).

[2] J. H. Chu and Lin I, Phys. Rev. Lett. **72**, 4009 (1994); H. Thomas, G. E. Morfill, V. Demmel, J. Goree, B. Feuerbacher, and D. Möhlmann, *ibid.* **73**, 652 (1994); Y. Hayashi and K. Tachibana, Jpn. J. Appl. Phys. **33**, L804 (1994); A. Melzer, T. Trottenberg, and A. Piel, Phys. Lett. A **191**, 301 (1994).

[3] H. M. Thomas and G. E. Morfill, Nature (London) **379**, 806 (1996).

[4] A. Melzer, A. Homann, and A. Piel, Phys. Rev. E **53**, 2757 (1996).

[5] A. Melzer, V. A. Schweigert, I. V. Schweigert, A. Homann, S. Peters, and A. Piel, Phys. Rev. E **54**, R46 (1996).

[6] S. Peters, A. Homann, A. Melzer, and A. Piel, Phys. Lett. A **223**, 389 (1996).

[7] J. B. Pieper and J. Goree, Phys. Rev. Lett. **77**, 3137 (1996).

[8] H. Totsuji, T. Kishimoto, and C. Totsuji, Phys. Rev. Lett. **78**, 3113 (1997); N. Otani and A. Bhattacharjee, Phys. Rev. Lett. **78**, 1468 (1997).

[9] S. V. Vladimirov and M. Nambu, Phys. Rev. E **52**, 2172 (1995).

[10] S. V. Vladimirov and O. Ishihara, Phys. Plasmas **3**, 444 (1996).

[11] O. Ishihara and S. V. Vladimirov, Phys. Plasmas **4**, 69 (1997).

[12] F. F. Chen, *Introduction to Plasma Physics* (Plenum, New York, 1974), Chap. 8.

[13] O. Ishihara, I. Alexeff, H. J. Doucet, and W. D. Jones, Phys. Fluids **21**, 2211 (1978).
[14] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966), p. 102.
[15] M. Nambu and H. Akama, Phys. Fluids **28**, 2300 (1985).
[16] F. Melandsø, Phys. Plasmas **3**, 3890 (1996).
[17] S. V. Vladimirov, P. V. Shevchenko, and N. F. Cramer, Phys. Rev. E **56**, 74 (1997); Phys. Plasmas **5**, 4 (1998).
[18] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953).
[19] M. Kitagawa and Y. H. Ohtsuki, Phys. Rev. B **16**, 5321 (1977).
[20] L. Mitchell (private communication).